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Geometry and dynamics of higher-spin frame fields

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Abstract: We give a systematic account of unconstrained free bosonic higher-spin fields on D-dimensional Minkowski and (Anti-)de Sitter spaces in the frame formalism. The generalized spin connections are determined by solving a chain of torsion-like constraints. Via a generalization of the vielbein postulate these allow to determine higher-spin Christoffel symbols, whose relation to the de Wit-Freedman connections is discussed. We prove that the generalized Einstein equations, despite being of higher-derivative order, give rise to the AdS Frønsdal equations in the compensator formulation. To this end we derive Damour-Deser identities for arbitrary spin on AdS. Finally we discuss the possibility of a geometrical and local action principle, which is manifestly invariant under unconstrained higher-spin symmetries.

Keywords: Space-Time Symmetries, Gauge Symmetry, M-Theory, Classical Theories of Gravity.

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1. Introduction

In the ongoing search for the underlying principles and symmetries of string/M-theory, the analysis of higher-spin (HS) theories has played a distinguished role [1 − 8]. Since the original work of Frønsdal [9, 10] in the 1970's there has been substantial progress in the following two directions. On the one hand, due to the work of Vasiliev, the problem of constructing consistently interacting HS theories coupled to gravity has been efficiently attacked [11 − 13] (see also [14] and references therein). On the other hand, the free higher-spin theories on Minkowski and AdS have been recently reformulated in a much more geometrical fashion, using 'unconstrained fields' (a term to be defined below), leading to manifestly HS invariant Einstein-like equations [15 − 18].

However, there is a certain tension between these two directions in the way they treat the HS fields. The approach of Vasiliev to construct interacting HS theories is based on the gauging of certain HS algebras in a similar fashion to gravity and supergravity theories. Like in supergravity this requires a frame-like formulation of HS fields, which generalizes the vielbein formalism of general relativity. Such a formalism has been developed by Vasiliev in [19]. (For a related approach see [20, 21].) It describes a spin-s field by a 1-form $e_\mu{}^{a_1\cdots a_{s-1}}$ carrying $s-1$ totally symmetric frame indices, together with a generalized spin connection $\omega_\mu{}^{a_1\cdots a_{s-1},b}$. In contrast, the recent progress in the geometrical formulation of free HS theories is based on a 'metric-like' formulation, in which a spin-s field is described by totally symmetric covariant tensor $h_{\mu_1\cdots\mu_s}$ of rank s, generalizing the metric tensor of Einstein gravity. However, in the latter approach it has been clear since the early work of de Wit and Freedman [22] that the natural object, replacing the Christoffel symbol of Riemannian geometry, is not a single connection but instead a hierarchy of $s-1$ connections, each being expressed as derivatives of the previous one. The top connection is identified with the HS Riemann tensor, which in turn is an s-derivative object of the physical spin-s field.

Until recently it has, accordingly, been unclear, how to take advantage of this appealing geometrical structure in that there seems to be no natural way to get standard 2$^{\text{nd}}$ order field equations. In order to recover the 2$^{\text{nd}}$ order spin-s equations formulated by Frønsdal one has to work with 'constrained' objects, in which the transformation parameter of the HS gauge symmetry is restricted to be traceless. This has the consequence that the trace of the 2$^{\text{nd}}$ de Wit-Freedman connection is invariant under the constrained HS symmetry. Consequently, it gives rise to the Frønsdal equations without employing the higher connections. Though this invariance under constrained HS transformations is sufficient to decouple the unphysical longitudinal degrees of freedom of a massless spin-s state, the constrained formulation seems to be unnatural once the coupling to gravity is considered since then, ultimately, the metric and thus the operation of 'taking the trace' becomes dynamical. Therefore an unconstrained formulation is clearly desirable.

Progress into this direction has been achieved due to the work of Francia and Sagnotti [15, 16]. (Apart from that, unconstrained HS fields appeared in [23 – 28], but also in string field theory, see [29, 5, 30] and references therein.) Their formulation is entirely geometrical in that it is given in terms of the trace-full, i.e. unconstrained, HS Riemann tensor. It requires, however, *non-local* field equations, in which inverse powers of the Laplacian appear in order to get rid of the higher derivative order. On the other hand, there exists an unconstrained local formulation, which handles the non-invariance of the Frønsdal equations under trace-full HS transformations via introducing so-called compensator fields [31, 32]. Unfortunately, this formulation is not geometrical anymore in the sense that the compensators are unrelated to the connections or the curvature tensor. However, it has been shown by Bekaert and Boulanger in [17, 18], based on [33], how to obtain in flat space a theory which is, first, purely geometrical and, second, local but still equivalent to the 2$^{\text{nd}}$ order Frønsdal equations. For this they proved that the generalized Einstein equations can be locally integrated via the Poincaré lemma, giving rise to the 2$^{\text{nd}}$ order compensator formulation of [15, 16], with the compensator fields appearing as 'integration constants'.

So on the one hand, the most geometrical formulation for metric-like fields seems to require the full chain of de Wit-Freedman connections and higher-derivative field equations. On the other hand, the frame-like formulation for free HS fields possesses only a single spin-connection-like object. However, in HS gauge theories (based on the associated HS algebras) higher connections do appear. These are the so-called 'extra fields'. But usually they are treated on a different footing than the lowest connection, since they can be decoupled at the free field level [34, 35]. This is usually done precisely to guarantee 2^{nd} order free field equations. But this is only possible if one works with trace-less tensors (i.e. with SO(D) instead of $GL(D)$ Young tableaux), in accordance with the fact that the higher de Wit-Freedman connections are only required in the trace-full, i.e. unconstrained case.

An extension of the Vasiliev formulation to unconstrained objects has been initiated in [36]. It has been shown for a spin-3 field on Minkowski space that also in this formalism the Frønsdal equations can be obtained from geometrical higher-derivative equations. More recently, this spin-3 analysis has been extended to AdS in [37]. However, in spite of these partial results (see also [18]), a complete account of an unconstrained frame formalism for arbitrary spin on AdS backgrounds, generalizing that of [19], is so far lacking. Instead the more general, but also more abstract technique of the so-called σ^--cohomology has been developed, which allows to analyze the dynamical content of HS theories to a large extent without requiring explicit formulas of the type we are going to consider in this paper (for an introduction in the unconstrained case see [14]). We think, however, that a more explicit treatment will be crucial for concrete applications. This leads us to reexamine the geometry of HS frame fields in a systematic and self-contained fashion.

More specifically, the paper is organized as follows. In section 2 we review linearized gravity and the metric-like formulation of free HS fields in flat space together with the de Wit-Freedman connections. The frame-like formalism of HS geometry will be discussed in section 3. We derive the torsion constraints, solve them explicitly and discuss the resulting Bianchi identities. Finally, we discuss the relation to the de Wit-Freedman connections. In section 4 we analyze the dynamics of HS fields, and we show that the Einstein equations give rise to the compensator form of the Frønsdal equations. For that purpose we derive a generalization of the Damour-Deser identity to AdS. We close with a brief discussion concerning the possibility of an action principle. Our conventions, the technical details related to the solutions of the torsion constraints and the proof of the Damour-Deser identity are contained in the appendices A–C.

2. Linearized gravity and higher-spin fields

2.1 Christoffel and spin connection for spin-2

One way to think about HS theories is as a generalization of Einstein gravity, in which the spin-2 metric tensor is replaced by a higher rank tensor. So, in order to set the stage for our later examinations, let us first recall the spin-2 case.

The Riemannian geometry underlying Einstein's theory can be formulated either in terms of the metric $g_{\mu\nu}$ or a frame field ('vielbein') ${e_\mu}^a$. Since we will later on focus on HS fields that generalize the frame formulation of gravity, we start by reviewing the vielbein formalism. At the purely kinematical level this formalism allows a gauge theoretical interpretation in the sense that the vielbein and the spin connection are viewed as components of a gauge connection, which is associated to the (Anti-)de Sitter or Poincaré algebra.[1] We will focus on the AdS case, but keep the cosmological constant explicit such that the Inönü-Wigner contraction $\Lambda \to 0$ can always be performed. The Lie algebra of the required AdS group $\mathrm{SO}(D-1,2)$ in D dimensions is spanned by generators $M_{AB} = -M_{BA}$ and reads

$$\begin{aligned} [M_{AB}, M_{CD}] &= \eta_{BC} M_{AD} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC} + \eta_{AD} M_{BC} \\ &\equiv f_{AB,CD}{}^{EF} M_{EF} \,, \end{aligned} \tag{2.1}$$

where

$$\eta_{AB} = \mathrm{diag}(-1,1,1,1,1,-1)\,, \qquad f_{AB,CD}{}^{EF} \;=\; 4\delta^{[E}_{[A}\eta_{B][C}\delta^{F]}_{D]} \,. \tag{2.2}$$

(For our conventions see appendix A.) Next we need to split this basis into a Lorentz covariant form, i.e.

$$\begin{aligned} [M_{ab}, M_{cd}] &= \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} + \eta_{ad} M_{bc} \,, \\ [M_{ab}, P_c] &= -2\eta_{c[a} P_{b]} \,, \qquad [P_a, P_b] \;=\; \Lambda M_{ab} \,, \end{aligned} \tag{2.3}$$

where we have defined $P_a = \sqrt{\Lambda} M_{a0'} = M_{a0'}/L$, with the AdS length L. Then writing the associated gauge field as $A_\mu = e_\mu{}^a P_a + \frac{1}{2}\omega_\mu{}^{ab} M_{ab}$, we obtain the required identification of vielbein and spin connection. Moreover, the torsion and curvature tensors naturally appear as the non-abelian field strengths of (2.1). Explicitly, one has

$$F \;=\; dA + A \wedge A \;=\; T^a P_a + \tfrac{1}{2}\mathcal{R}^{ab} M_{ab} \,, \tag{2.4}$$

which indeed reads

$$T^a \;=\; D^L e^a \;\equiv\; de^a + \omega^a{}_b \wedge e^b \,, \tag{2.5}$$

$$\mathcal{R}^{ab} \;=\; R^{ab} + \Lambda e^a \wedge e^b \;\equiv\; d\omega^{ab} + \omega^{ac} \wedge \omega_c{}^b + \Lambda e^a \wedge e^b \,. \tag{2.6}$$

Imposing the condition $T^a = 0$ allows to solve for the spin connection $\omega_\mu{}^{ab}$ in terms of derivatives of the vielbein. Inserting this into $R_{\mu\nu}{}^{ab}$ gives rise to the standard Riemann tensor as a function of second derivatives of the metric.

Let us now turn to the metric formulation, in which the Levi-Civita connection is encoded in the Christoffel symbols. To see where these enter, we first note that the Lorentz covariant derivative $D^L_\mu e_\nu{}^a$ of the vielbein is not covariant under diffeomorphisms, in contrast to the antisymmetric part $D^L_{[\mu} e_{\nu]}{}^a$, which is the torsion 2-form defined in (2.5). So in order to define a derivative which is covariant with respect to local Lorentz transformations and diffeomorphisms, we have to introduce a connection which is symmetric in μ,ν. These are the Christoffel symbols, related to the spin connection via the metricity condition ('vielbein postulate')

$$D_\mu e_\nu{}^a \;=\; \partial_\mu e_\nu{}^a + \omega_\mu{}^{ab} e_{\nu b} - \Gamma^\rho_{\mu\nu} e_\rho{}^a \;=\; 0 \,. \tag{2.7}$$

[1] However, as far as the dynamics is concerned, this interpretation breaks down, except in special cases like three-dimensional gravity [38] or specific forms of Lovelock gravities in odd dimensions [39, 40], which can be viewed as Yang-Mills gauge theories based on Chern-Simons forms. In connection to HS and Kaluza-Klein theories see [37] and [41], respectively.

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In order to compare with the free HS dynamics to be discussed below, we have to consider linearized gravity. By linearizing around Minkowski space, $e_\mu{}^a = \delta_\mu{}^a + \kappa h_\mu{}^a$, the metricity condition (2.7) reads at the first non-trivial order

$$\partial_\mu h_{\nu\rho} + \omega_{\mu|\rho,\nu} - \Gamma_{\rho,\mu\nu} \; = \; 0 \; , \tag{2.8}$$

in which the distinction between flat and curved indices is now redundant and will therefore henceforth be ignored.[2] At this stage, the frame formulation has the basic consequence that $h_{\mu\nu}$ is not symmetric, but also has an antisymmetric part. Accompanied with this is an additional gauge symmetry, namely the local Lorentz symmetry. In the linearization the latter reads together with the diffeomorphisms

$$\delta_\xi h_{\mu\nu} \; = \; \partial_\mu \xi_\nu - \Lambda_{\nu,\mu} \; , \tag{2.9}$$

i.e. the Lorentz transformations act as Stückelberg symmetries with anti-symmetric shift parameter $\Lambda_{\mu,\nu}$, while the connections transform as

$$\begin{aligned} \delta_\Lambda \omega_{\mu|\nu,\rho} = \partial_\mu \Lambda_{\nu,\rho} \; , \qquad \delta_\xi \omega_{\mu|\nu,\rho} \; &= \; 0 \; , \\ \delta_\xi \Gamma_{\rho,\mu\nu} = \partial_\mu \partial_\nu \xi_\rho \; , \qquad \delta_\Lambda \Gamma_{\rho,\mu\nu} \; &= \; 0 \; . \end{aligned} \tag{2.10}$$

Using the explicit expression for the spin connection obtained from (2.5),

$$\omega_{\mu|\nu,\rho} \; = \; \partial_\mu h_{[\nu\rho]} + \partial_\rho h_{(\mu\nu)} - \partial_\nu h_{(\mu\rho)} \; , \tag{2.11}$$

which depends also on the antisymmetric part, one derives from (2.8) for the Christoffel symbols[3]

$$\Gamma_{\rho,\mu\nu} \; = \; \partial_\mu h_{(\rho\nu)} + \partial_\nu h_{(\rho\mu)} - \partial_\rho h_{(\mu\nu)} \; . \tag{2.12}$$

Note that here the antisymmetric part drops out, in agreement with the fact that $\Gamma_{\rho,\mu\nu}$ is Lorentz gauge-invariant. These shift-symmetries can in turn be used to fix a gauge, in which the anti-symmetric part of $h_{\mu\nu}$ is gauged away and only the symmetric part survives. As this gauge-fixing would be violated by generic diffeomorphisms ξ^ρ, this requires a compensating Lorentz transformation with parameter $\Lambda_{\nu,\mu} = \partial_{[\mu}\xi_{\nu]}$, giving rise to the standard diffeomorphism symmetry on $h_{\mu\nu}$,

$$\delta_\xi h_{\mu\nu} \; = \; \partial_{(\mu}\xi_{\nu)} \; , \qquad \delta_\xi \omega_{\mu|\nu,\rho} \; = \; \partial_\mu \partial_{[\rho}\xi_{\nu]} \; . \tag{2.13}$$

Here, after gauge-fixing, also the spin connection transforms non-trivially under diffeomorphisms.

[2]Here, we use the notation $\omega_{\mu|\rho,\nu}$, in which a comma separates indices in different rows of a Young-tableau, in order to comply with the notation used for HS fields later on.

[3]The reader might miss a factor of $\frac{1}{2}$, which is due to the chosen normalization in the expansion of $e_\mu{}^a$ around flat space.

2.2 Higher-spin fields and de Wit-Freedman connections

After reviewing the spin-2 case, we will now discuss HS fields. These are given by totally symmetric tensor fields $h_{\mu_1\cdots\mu_s}$ of rank s, which generalize the metric tensor of Einstein gravity. In the massless case their dynamics has to permit a gauge symmetry, which eliminates the unphysical longitudinal degrees of freedom and generalizes the diffeomorphism symmetry of general relativity. It is parameterized by a symmetric rank $s-1$ tensor $\epsilon_{\mu_1\cdots\mu_{s-1}}$ and reads

$$\delta_\epsilon h_{\mu_1\cdots\mu_s} \;=\; \partial_{(\mu_1}\epsilon_{\mu_2\cdots\mu_s)}\;. \tag{2.14}$$

An action which generalizes the linearized Einstein-Hilbert term to higher spin and stays invariant under (2.14) has been given by Frønsdal [9]. It can be written as

$$\begin{aligned} S_{\rm F}[h] = \frac{1}{2}\int d^Dx\big(&\partial_\mu h_{\nu_1\cdots\nu_s}\partial^\mu h^{\nu_1\cdots\nu_s} - \tfrac{1}{2}s(s-1)\partial_\mu h'_{\nu_3\cdots\nu_s}\partial^\mu h'^{\nu_3\cdots\nu_s} \\ &+s(s-1)\partial_\mu h'_{\nu_3\cdots\nu_s}\partial\cdot h^{\mu\nu_3\cdots\nu_s} - s\partial\cdot h_{\nu_2\cdots\nu_s}\partial\cdot h^{\nu_2\cdots\nu_s} \\ &-\tfrac{1}{4}s(s-1)(s-2)\partial\cdot h'_{\nu_1\cdots\nu_{s-3}}\partial\cdot h'^{\nu_1\cdots\nu_{s-3}}\big)\;. \end{aligned} \tag{2.15}$$

Here h' denotes the trace in the Minkowski metric and $\partial\cdot h_{\nu_2\cdots\nu_s} = \partial^\rho h_{\rho\nu_2\cdots\nu_2}$. The action (2.15) is invariant under (2.14), provided the transformation parameter is traceless. Moreover, the double trace of h has to be set to zero in order for the field to describe the correct number of degrees of freedom [22]. This is the so-called constrained formulation, for which in total

$$\epsilon'_{\nu_3\cdots\nu_{s-1}} \;=\; \epsilon^\rho{}_{\rho\nu_3\cdots\nu_{s-1}} \;=\; 0\;, \qquad h''_{\rho_5\cdots\rho_s} \;=\; h^{\mu\nu}{}_{\mu\nu\rho_5\cdots\rho_s} \;=\; 0\;. \tag{2.16}$$

Note that the double-tracelessness constraint stays invariant under the constrained HS symmetries. The spin-s field equations derived from (2.15) read

$$\mathcal{F}_{\mu_1\cdots\mu_s} \;=\; \Box h_{\mu_1\cdots\mu_s} - s\partial_{(\mu_1}\partial\cdot h_{\mu_2\cdots\mu_s)} + \frac{s(s-1)}{2}\partial_{(\mu_1}\partial_{\mu_2}h'_{\mu_3\cdots\mu_s)} \;=\; 0\;, \tag{2.17}$$

which defines the Frønsdal operator $\mathcal{F}$.

This formulation of HS dynamics can be extended to AdS backgrounds. First, the minimal substitution $\partial_\mu \to \nabla_\mu$ in the action (2.15) and the symmetry variations (2.14), with ∇_μ denoting the AdS covariant derivative, violates the HS invariance. For maximally symmetric backgrounds this can be compensated by adding a mass-like term proportional to the cosmological constant [10]. (For recent results on HS fields on AdS in the partially massless and massive case see [42–45].) Then the equations of motion are

$$\begin{aligned} \mathcal{F}^{\rm AdS}_{\mu_1\cdots\mu_s} &= \Box h_{\mu_1\cdots\mu_s} - s\nabla_{(\mu_1}\nabla\cdot h_{\mu_2\cdots\mu_s)} + \frac{s(s-1)}{2}\nabla_{(\mu_1}\nabla_{\mu_2}h'_{\mu_3\cdots\mu_s)} \\ &\quad -\Lambda\Big(\big((D-3+s)(s-2)-s\big)h_{\mu_1\cdots\mu_s} + s(s-1)g_{(\mu_1\mu_2}h'_{\mu_3\cdots\mu_s)}\Big) \;=\; 0\;. \end{aligned} \tag{2.18}$$

Its variation under unconstrained HS transformations reads

$$\delta_\epsilon\mathcal{F}^{\rm AdS}_{\mu_1\cdots\mu_s} \;=\; \frac{(s-1)(s-2)}{2}\Big(\nabla_{(\mu_1}\nabla_{\mu_2}\nabla_{\mu_3}\epsilon'_{\mu_4\cdots\mu_s)} - 4\Lambda g_{(\mu_1\mu_2}\nabla_{\mu_3}\epsilon'_{\mu_4\cdots\mu_s)}\Big)\;, \tag{2.19}$$

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and so, indeed, it is only invariant if $\epsilon' = 0$.

The Frønsdal action (2.15) together with the constraints (2.16) consistently describes the free propagation of a massless spin-s field. However, as it stands, the formulation is not very geometrical, since (2.15) has been determined by hand, and there is no obvious way to rewrite it in terms of HS curvatures. This is in contrast to the spin-2 case, for which (2.15) coincides with the linearized Einstein-Hilbert action, and can therefore be written in a manifestly spin-2, that is, diffeomorphism invariant form. A first step towards a more geometrical formulation of HS fields would accordingly require a generalization of the Christoffel symbols of Riemannian geometry in an unconstrained way. An appealing formalism has been presented already by de Wit and Freedman in [22] for the flat case, and will be reviewed in the following. (It has only been possible recently to tackle the AdS analogue, at least perturbatively in the inverse AdS radius [46].)

To start with, we have to find Christoffel symbols, which are first order in derivatives of the HS fields and transform as connections. The definition[4]

$$\Gamma^{(1)}_{\rho,\mu_1\cdots\mu_s} \;=\; -\partial_\rho h_{\mu_1\cdots\mu_s} + s\partial_{(\mu_1} h_{\mu_2\cdots\mu_s)\rho}\;, \tag{2.20}$$

gives rise to the transformation behaviour

$$\delta_\epsilon\Gamma^{(1)}_{\rho,\mu_1\cdots\mu_s} \;=\; (s-1)\partial_{(\mu_1}\partial_{\mu_2}\epsilon_{\mu_3\cdots\mu_s)\rho}\;. \tag{2.21}$$

This transformation is as simple as possible, since the relative coefficients in (2.20) have been chosen such that the index ρ appears in (2.21) only on the transformation parameter. Moreover, for spin-2 it reduces to the standard Christoffel symbol (2.12). However, the transformation (2.21) is not truly connection-like, since it does not allow for the definition of an invariant curvature tensor in terms of derivatives of $\Gamma^{(1)}_{\rho,\mu_1\cdots\mu_s}$. Instead, it allows to define a hierarchy of s connection-like objects, each of which being defined as derivatives of the previous one via the recursion relation ($1 \leq m \leq s$)

$$\Gamma^{(m)}_{\rho_1\cdots\rho_m,\mu_1\cdots\mu_s} \;=\; \partial_{\rho_1}\Gamma^{(m-1)}_{\rho_2\cdots\rho_m,\mu_1\cdots\mu_s} - \frac{s}{m}\partial_{(\mu_1}\Gamma^{(m-1)}_{|\rho_2\cdots\rho_m,\rho_1|\mu_2\cdots\mu_s)}\;. \tag{2.22}$$

These are totally symmetric in the two sets of indices μ_i and ρ_i, although for the latter it is not manifest. They transform under the HS symmetry as

$$\delta_\epsilon\Gamma^{(m)}_{\rho_1\cdots\rho_m,\mu_1\cdots\mu_s} \;=\; (-1)^{m+1}\binom{s-1}{m}\partial_{(\mu_1}\cdots\partial_{\mu_{m+1}}\epsilon_{\mu_{m+2}\cdots\mu_s)\rho_1\cdots\rho_m}\;. \tag{2.23}$$

Therefore only the $(s-1)$-connection transforms as a proper connection in the sense that

$$\delta_\epsilon\Gamma^{(s-1)}_{\rho_1\cdots\rho_{s-1},\mu_1\cdots\mu_s} \;=\; (-1)^s\partial_{\mu_1}\cdots\partial_{\mu_s}\epsilon_{\rho_1\cdots\rho_{s-1}}\;. \tag{2.24}$$

Consequently, the top 'connection' in (2.22) is actually an invariant curvature tensor defined in terms of derivatives of $\Gamma^{(s-1)}$,

$$R_{\rho_1\cdots\rho_s,\mu_1\cdots\mu_s} \;=\; \Gamma^{(s)}_{\rho_1\cdots\rho_s,\mu_1\cdots\mu_s}\;. \tag{2.25}$$

[4]Here we have chosen a different overall sign than in [22].

So there exists a geometrical structure for unconstrained HS fields, which naturally extends the known spin-1 and spin-2 cases. In particular, the gauge-invariant curvatures are s-derivative objects, generalizing the $1^{\rm st}$ order field strength of electrodynamics and the $2^{\rm nd}$ order Riemann tensor of general relativity. However, due to this higher-derivative nature it is not clear how to obtain sensible $2^{\rm nd}$ order field equations. This is where the constrained formulation enters. In fact, once we constrain as in (2.16), the trace of the second de Wit-Freedman connection $\Gamma^{(2)}$ turns out to be invariant, since it transforms only into the trace part of ϵ, as can be seen from (2.23). Invariant $2^{\rm nd}$ order field equations can then be written as

$$\Gamma^{(2)\rho}{}_{\rho,\mu_1\cdots\mu_s} \;=\; 0\ , \tag{2.26}$$

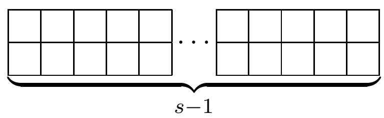
which coincide with the Frønsdal equations — as it should be by gauge invariance. More recently it has been shown that the generalized Einstein equation, stating that the trace of the HS Riemann tensor (2.25) vanishes, effectively also gives rise to $2^{\rm nd}$ order field equations through local integrations [17, 18]. This analysis will be extended to the frame formalism and AdS later on.

3. Higher-spin geometry

In this section we start the analysis of HS fields in the frame formalism. We introduce the corresponding Lie algebra, which determines the HS gauge symmetries, and whose field strengths will be interpreted as HS torsion and curvature tensors, respectively. After imposing torsion constraints we give their explicit solutions and discuss the resulting Bianchi identities together with the relation to the de Wit-Freedman connections.

3.1 Higher-spin gauge algebra and connections

We are going to consider the free dynamics of HS fields on AdS. Therefore, we have to fix the spin-2 vielbein to the background value $\bar{e}_\mu{}^a$ of the AdS geometry, being covariantly characterized by $\bar{\mathcal{R}}^{ab}=0$. In order to introduce the HS frame fields, we extend the Lie algebra $\mathfrak{so}(D-1,2)$ in (2.1) for each spin-s field by a generator $Q_{A_1\cdots A_{s-1},B_1\cdots B_{s-1}}$, which lives in the two-row Young tableaux

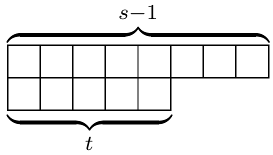
$Q_{A(s-1),B(s-1)}$:  . (3.1)

Here and in the following we use the convenient short-hand notation $A(s)=A_1\cdots A_s$ for totally symmetric indices and similarly for Lorentz indices. The commutation relations with the spin-2 or $\mathfrak{so}(D-1,2)$ generators are entirely fixed by representation theory and read

$$\begin{aligned}
[M_{AB},Q_{C(s-1),D(s-1)}] &= -4(s-1)\eta_{\underbrace{A\langle C_{s-1}}Q_{|\underbrace{B}|C(s-2),D(s-1)\rangle}} \\
&= -2\Big(\eta_{\underbrace{AC_1}}Q_{\underbrace{B}C_2\cdots C_{s-1},D(s-1)}+\eta_{\underbrace{AC_2}}Q_{\underbrace{C_1B}C_3\cdots C_{s-1},D(s-1)}+\cdots \\
&\qquad +\eta_{\underbrace{AD_{s-1}}}Q_{\underbrace{C(s-1),D_1\cdots D_{s-2}B}}\Big)\ .
\end{aligned} \tag{3.2}$$

Here we introduced projectors according to the Young symmetries imposed by the left-hand side. (For our conventions see appendix A.) As in the spin-2 case above we are next splitting into a Lorentz covariant basis, for which the generators $Q_{A(s-1),B(s-1)}$ decompose into $Q_{a(s-1),b(t)}$ for $0 \leq t \leq s-1$, being in the Young tableau

$Q_{a(s-1),b(t)}$:



The commutation relations (3.2) then read

$$\begin{aligned} [M_{ab}, Q_{c(s-1),d(s-1)}] &= -4(s-1)\eta_{a\langle \underbrace{c_{s-1} Q_{|b|}}c(s-2),d(s-1)\rangle} , \\ [M_{ab}, Q_{c(s-1),d(t)}] &= -2\big((s-1)\eta_{a\langle \underbrace{c_{s-1} Q_{|b|}}c(s-2),d(t)\rangle} + t\eta_{a\langle \underbrace{d_t Q_{c(s-1),|b|}}d(t-1)\rangle}\big) , \\ [P_a, Q_{c(s-1),d(t)}] &= t(s-t+1)\frac{s-1}{s-t}\eta_{a\langle c_{s-1}} Q_{c(s-2)d_t,d(t-1)\rangle} \\ &\quad - (s-t-1)\Lambda Q_{c(s-1),d(t)a} , \end{aligned} \tag{3.3}$$

where the brackets $\langle , \rangle$ impose the the corresponding Young projections. To keep track of the powers of Λ, dimensional analysis is useful. By defining the (mass) dimensions $[\Lambda] = [L^{-2}] = 2$ and $[M_{AB}] = 0$ it follows that $[P_a] = 1$ and $[Q_{a(s-1),b(t)}] = s-1-t$. The Lie brackets $[Q, Q]$ vanish, since these would correspond to self-interactions of the HS fields, which do not enter the free dynamics. In fact, the algebra (3.2) spanned by M_{AB} and the Q can be viewed as a truncation of the infinite-dimensional HS algebras (sometimes denoted by $\mathfrak{ho}(D-1,2)$) of Vasiliev [47, 14], which appear upon linearization [36, 37].

Next we are introducing a HS gauge field $\mathcal{A} = \mathcal{A}_\mu dx^\mu$, which reads in components

$$\mathcal{A}_\mu \;=\; \bar{e}_\mu{}^a P_a + \frac{1}{2}\bar{\omega}_\mu{}^{ab} M_{ab} + \sum_{s=3}^{\infty} \mathcal{W}_\mu^{(s)} , \tag{3.4}$$

where the spin-s contribution is given by

$$\mathcal{W}_\mu^{(s)} = \frac{1}{(s-1)!} e_\mu{}^{a(s-1)} Q_{a(s-1)} + \sum_{t=1}^{s-1} \frac{s-t}{s!t!} \omega_\mu{}^{a(s-1),b(t)} Q_{a(s-1),b(t)} . \tag{3.5}$$

The coefficients $(s-t)/(s!t!)$ impose unit-strength normalizations and follow from the Hook length formula (see, e.g., [50]) for an $(s-1,t)$ Young tableau. The fields $\omega_\mu{}^{a(s-1),b(t)}$ will be referred to in the following as HS connections, while the fields $e_\mu{}^{a(s-1)}$ will later on be identified with the physical spin-s fields, or the generalized vielbeins. We define the canonical dimension $[\mathcal{A}] = 1$, from which it follows that $[e_\mu{}^{a(s-1)}] = 2-s$ and $[\omega_\mu{}^{a(s-1),b(t)}] = 2-s+t$. In order to derive HS torsions and curvatures we compute as above the non-abelian HS field strength, which is given by

$$\mathcal{F}_{\mu\nu} \;=\; \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \;=\; \bar{T}_{\mu\nu}{}^a P_a + \frac{1}{2}\bar{\mathcal{R}}_{\mu\nu}{}^{ab} M_{ab} + \sum_{s=3}^{\infty} \mathcal{F}_{\mu\nu}^{(s)} , \tag{3.6}$$

where the spin-s curvatures read

$$\mathcal{F}^{(s)}_{\mu\nu} = \sum_{t=0}^{s-2} \frac{s-t}{s!t!} T_{\mu\nu}{}^{a(s-1),b(t)} Q_{a(s-1),b(t)} + \frac{1}{s!(s-1)!} R_{\mu\nu}{}^{a(s-1),b(s-1)} Q_{a(s-1),b(s-1)} \ . \tag{3.7}$$

By using (3.3) we find

$$\begin{aligned} R_{\mu\nu}{}^{a(s-1),b(s-1)} &= \bar{D}_\mu \omega_\nu{}^{a(s-1),b(s-1)} - 2(s-1)\Lambda \omega_\nu{}^{\langle a(s-1),b(s-2)} \bar{e}_\mu{}^{b_{s-1}\rangle} - (\mu \leftrightarrow \nu) \,, \\ T_{\mu\nu}{}^{a(s-1),b(t)} &= \bar{D}_\mu \omega_\nu{}^{a(s-1),b(t)} - \omega_\nu{}^{a(s-1),b(t)c} \bar{e}_{\mu c} \\ & \quad - t(s-t+1)\Lambda \omega_\nu{}^{\langle a(s-1),b(t-1)} \bar{e}_\mu{}^{b_t\rangle} - (\mu \leftrightarrow \nu) \,, \end{aligned} \tag{3.8}$$

where it is understood that $\omega_\mu{}^{a(s-1)} \equiv e_\mu{}^{a(s-1)}$. Here we defined Lorentz covariant derivatives, which are given by

$$\begin{aligned} \bar{D}_\mu \omega_\nu{}^{a(s-1),b(t)} &= \partial_\mu \omega_\nu{}^{a(s-1),b(t)} + (s-1)\bar{\omega}_\mu^{\langle a_{s-1}|a|} \omega_{\nu a}{}^{a(s-2),b(t)\rangle} \\ & \quad + t\, \bar{\omega}_\mu{}^{\langle b_t|a|} \omega_\nu{}^{a(s-1),}{}_a{}^{b(t-1)\rangle} \ . \end{aligned} \tag{3.9}$$

Finally, we give the HS gauge transformations, under which the field strengths above stay invariant. Defining the transformation parameter to be

$$\epsilon \ = \ \frac{1}{(s-1)!} \epsilon^{a(s-1)} Q_{a(s-1)} + \sum_{t=1}^{s-t} \frac{s-1}{s!t!} \epsilon^{a(s-1),b(t)} Q_{a(s-1),b(t)} \ , \tag{3.10}$$

the gauge variation $\delta\mathcal{W}_\mu = D_\mu \epsilon = \partial_\mu \epsilon + [\mathcal{W}_\mu, \epsilon]$ of the various components reads

$$\begin{aligned} \delta_\epsilon e_\mu{}^{a(s-1)} &= \bar{D}_\mu \epsilon^{a(s-1)} - \epsilon^{a(s-1),c} \bar{e}_{\mu c} \,, \\ \delta_\epsilon \omega_\mu{}^{a(s-1),b(t)} &= \bar{D}_\mu \epsilon^{a(s-1),b(t)} - \epsilon^{a(s-1),b(t)c} \bar{e}_{\mu c} - t(s-t+1)\Lambda \epsilon^{\langle a(s-1),b(t-1)} \bar{e}_\mu{}^{b_t\rangle} . \end{aligned} \tag{3.11}$$

The parameter $\epsilon^{a(s-1)}$ corresponding to the lowest HS generator will later give rise to the 'physical' HS symmetry. In contrast, the higher symmetries given by $\epsilon^{a(s-1),b(t)}$ for $t \geq 1$ act as Stückelberg shift symmetries, that correspond to the linearized Lorentz transformations in (2.9) and which subsequently will be gauge-fixed.

3.2 Torsion constraints and their solutions

We now turn to the torsion constraints for the HS connections. As in the spin-2 case reviewed in section 2.1, we identify the field strengths associated to the lowest gauge fields (containing the physical HS field) with the torsion tensors, while the top component of the field strength will be identified with the HS generalization of the Riemannian curvature tensor, cf. the expansion in (3.7).

Specifically, we impose the constraints

$$T_{\mu\nu}{}^{a(s-1),b(t)} \ = \ 0 \,, \qquad 0 \leq t \leq s-2 \ . \tag{3.12}$$

These allow to solve for the connection $\omega_\mu{}^{a(s-1),b(t)}$ at level t in terms of the first derivative of the 'previous' connection $\omega_\mu{}^{a(s-1),b(t-1)}$. (This parallels the elimination of the so-called ‘extra fields’ in terms of the physical field carried out in the constrained formalism in [48, 49].) To this end we introduce HS coefficients of anholonomity, which read

$$\begin{aligned} \Omega^{bc|a(s-1)} &= \bar{e}^{\mu b}\bar{e}^{\nu c}\big(\bar{D}_\mu e_\nu{}^{a(s-1)} - \bar{D}_\nu e_\mu{}^{a(s-1)}\big)\,, \\ \Omega^{cd|a(s-1),b(t)} &= \bar{e}^{\mu c}\bar{e}^{\nu d}\big(\bar{D}_\mu \omega_\nu{}^{a(s-1),b(t)} + \Lambda t(s-t+1)\omega_\mu{}^{\langle a(s-1),b(t-1)}\bar{e}_\nu{}^{b_t\rangle} - (\mu\leftrightarrow\nu)\big)\,. \end{aligned} \tag{3.13}$$

The torsion constraints (3.12) can then be written as

$$\Omega^{cd|a(s-1),b(t)} \;=\; \omega^{d|a(s-1),b(t)c} - \omega^{c|a(s-1),b(t)d}\,, \tag{3.14}$$

where we converted the 1-form index on ω into a flat one. We start with the first constraint in (3.12), i.e. $t=0$, in order to determine the HS connection at $t=1$. One finds the general solution

$$\omega^{a|b(s-1),c} \;=\; \frac{1}{2}\left(\Omega^{ca|b(s-1)} + \Omega^{c(b_1|b_2\cdots b_{s-1})a} + \Omega^{a(b_1|b_2\cdots b_{s-1})c}\right) + \xi^{b(s-1),ac}\,. \tag{3.15}$$

Here $\xi^{b(s-1),ac}$ denotes a gauge degree of freedom. To be more precise, we note that for (3.15) to transform as a HS connection according to (3.11), we have to assign a non-trivial transformation behaviour to ξ:

$$\delta_\epsilon \xi_{b(s-1),ac} \;=\; \bar{D}_{\langle a}\epsilon_{b(s-1),c\rangle} - s\,\epsilon_{\langle b(s-1)}\eta_{ac\rangle} - \epsilon_{b(s-1),ac}\,. \tag{3.16}$$

In particular, we see that it is subject to the Stückelberg shift symmetry parametrized by $\epsilon_{b(s-1),ac}$. Therefore, by gauge-fixing this symmetry, ξ can be set to zero. In terms of the HS connection ω this can be interpreted as follows. A priori, ω takes values in the Young tableaux

$$\ydiagram{1} \otimes \overbrace{\ydiagram{5,1}}^{s-1} = \overbrace{\ydiagram{5,1,1}}^{s-1} \oplus \overbrace{\ydiagram{6,1}}^{s} \oplus \overbrace{\ydiagram{5,2}}^{s-1}\,. \tag{3.17}$$

The part in the $(s-1,2)$ tableau is precisely given by $\xi^{b(s-1),ac}$ and is not determined by the torsion constraint. As such it can be gauge-fixed to zero. Note that by (3.16) this requires a compensating gauge transformation with $\epsilon_{b(s-1),ac} = \bar{D}_{\langle a}\epsilon_{b(s-1),c\rangle} - s\,\epsilon_{\langle b(s-1)}\eta_{ac\rangle}$.

Before we turn to the higher connections we are going to perform another gauge-fixing. In fact, so far the HS field $e_\mu{}^{a(s-1)}$ carries irreducible parts according to the decomposition

$$\ydiagram{1} \otimes \underbrace{\ydiagram{2}\cdots\ydiagram{2}}_{s-1} = \underbrace{\ydiagram{2}\cdots\ydiagram{3}}_{s} \oplus \overbrace{\ydiagram{5,1}}^{s-1}\,, \tag{3.18}$$

i.e. it contains the completely symmetric physical part

$$h_{\mu_1\cdots\mu_s} := \bar{e}_{(\mu_1}^{\;a_1}\cdots\bar{e}_{\mu_{s-1}}^{\;a_{s-1}}e_{\mu_s)a_1\cdots a_{s-1}}\,, \tag{3.19}$$

but also the hook-like part. The latter can in turn be gauged away by fixing the symmetry spanned by $\epsilon_{a(s-1),b}$, cf. (3.11). The residual gauge symmetry on h is then given by

$$\delta_\epsilon h_{\mu_1\cdots\mu_s} = \nabla_{(\mu_1}\epsilon_{\mu_2\cdots\mu_s)}\,, \tag{3.20}$$

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in agreement with (2.14), but for AdS backgrounds.[5] After gauge-fixing, the first HS connection can be rewritten as

$$\omega^{a|b(s-1),c} \;=\; \frac{s}{s-1}\mathbb{P}_{(s-1,1)}\Omega^{ca|b(s-1)}\;. \tag{3.21}$$

The resulting connection then takes values only in the $(s,1)$ tableaux. For instance, in case of a spin-3 field on flat space one can show

$$\omega_{\mu|\nu(2),\rho} \;=\; \left(\tfrac{1}{2}\left(\mathbb{P}_{\begin{array}{|c|c|c|}\hline \mu&\nu_1&\rho\\ \hline \nu_2 \\ \cline{1-1}\end{array}} + \mathbb{P}_{\begin{array}{|c|c|c|}\hline \mu&\nu_2&\rho\\ \hline \nu_1 \\ \cline{1-1}\end{array}}\right) - \mathbb{P}_{\begin{array}{|c|c|c|}\hline \mu&\nu_1&\nu_2\\ \hline \rho \\ \cline{1-1}\end{array}}\right)\partial_\mu h_{\rho\nu(2)} \qquad .$$

It turns out that, in order to solve the torsion constraints for $t>1$, first the lower constraints have to be solved. This requirement comes about as follows: Provided the background geometry is AdS, the torsion constraint at level t implies a Bianchi identity, which in turn gives a condition on the HS coefficients of anholonomity. In the next section we will prove that this reads

$$\Omega^{[ab|c(s-1),d(t-1)|e]} = 0\;. \tag{3.22}$$

Moreover, after gauge-fixing this relation extends further in that antisymmetrization of a,b with any of the remaining indices gives zero. This allows to prove that the following expressions solve the torsion constraints:

$$\omega_{\mu|\nu(s-1),\rho(t)} \;=\; \frac{s-t+1}{s-t}\mathbb{P}_{(s-1,t)}\Omega_{\rho_t\mu|\nu(s-1),\rho(t-1)} + \xi_{\nu(s-1),\rho(t)\mu}\;. \tag{3.23}$$

Here ξ denotes again a gauge-degree of freedom. To be more precise, in analogy to (3.17), the irreducible parts of $\omega^{(s-1,t)}$ read

$$(1)\;\otimes\;(s-1,t)\;=\;(s,t)\;\oplus\;(s-1,t,1)\;\oplus\;(s-1,t+1)\;, \tag{3.24}$$

and ξ is the part in the $(s-1,t+1)$ Young tableau. It corresponds to the shift symmetry in this tableau, which is given by $\delta_\epsilon\xi_{a(s-1),b(t+1)} = -\epsilon_{a(s-1),b(t+1)}$. In order to solve for the HS connections at level $t+1$ in terms of the coefficients of anholonomity at level t, also this symmetry has to be gauge-fixed. This gives rise to the compensating transformation

$$\epsilon_{\nu(s-1),\rho(t+1)} \;=\; \nabla_{\langle\rho_1}\cdots\nabla_{\rho_{t+1}}\epsilon_{\nu(s-1))\rangle} + \mathcal{O}(\Lambda)\;, \tag{3.25}$$

which in turn expresses all transformation parameter in terms of the physical HS symmetry. After this gauge-fixing, only the (s,t) contribution in the decomposition (3.24) is non-zero. This follows from the fact that the antisymmetrization over three indices in $(s-1,t,1)$ vanishes identically due to the total symmetry of h and of the partial derivatives (see eq. (3.26) below).

[5] Here and in the following we denote the AdS-background covariant derivative acting only on curved indices by ∇_μ.

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To better understand (3.23) we first examine it on flat space, where it can be rewritten in a closed form in terms of derivatives of the physical HS fields:

$$\omega_{\mu|\nu(s-1),\rho(t)} \;=\; \frac{s}{s-t}\mathbb{P}_{(s-1,t)}\partial_{\rho(t)}h_{\nu(s-1)\mu} \;=\; \frac{s}{s-t}\partial_{\langle\rho(t)}h_{\nu(s-1)\rangle\mu}\ . \tag{3.26}$$

Writing out the projector explicitly one finds for $t=1$

$$\omega_{\mu|\nu(s-1),\rho} = \partial_\rho h_{\nu(s-1)\mu} - \partial_{(\nu_1}h_{\nu_2\cdots\nu_{s-1})\rho\mu}\ , \tag{3.27}$$

while the higher ones are given by

$$\omega_{\mu|\nu(s-1),\rho(t)} = \mathbb{P}_{\nu(s-1)}\mathbb{P}_{\rho(t)}\sum_{m=0}^{t}\binom{t}{m}(-1)^m\partial_{\rho(t-m)\nu(m)}h_{\nu_{m+1}\cdots\nu_{s-1}\rho_{t-m+1}\cdots\rho_t\mu}\ , \tag{3.28}$$

where $\mathbb{P}_{\nu(s-1)}$, etc. imposes unit-strength symmetrization.

Let us now inspect the AdS case more explicitly. Even though we have a closed expression in terms of the Ω in (3.23), it turns out to be much more tedious to give these in terms of (AdS-covariant) derivatives of the physical HS fields. While the first connection ($t=1$) coincides with the flat space expression upon a minimal substitution,

$$\omega_{\mu|\nu(s-1),\rho} \;=\; \nabla_\rho h_{\nu(s-1)\mu} - \nabla_{(\nu_1}h_{\nu_2\cdots\nu_{s-1})\mu}\ , \tag{3.29}$$

this is not so for $t\geq 2$. Indeed, there are correction terms proportional to Λ, which read for $t=2$

$$\omega_{\mu|\nu(s-1),\rho(2)} = \frac{s}{s-2}\nabla_{\langle\rho(2)}h_{\nu(s-1)\rangle\mu} - \Lambda\frac{s(s-1)}{s-2}g_{\langle\rho(2)}h_{\nu(s-1)\rangle\mu}\ . \tag{3.30}$$

In order to determine analogous expressions for all higher connections, one can prove a recursion relation by use of (3.23), whose partial solutions we give in appendix B. In total we have

$$\begin{aligned}\omega_{\mu|\nu(s-1),\rho(t)} \;=\; &\frac{s}{s-t}\nabla_{\langle\rho_1}\cdots\nabla_{\rho_t}h_{\nu(s-1)\rangle\mu}\\ &+\sum_{k=1}^{[t/2]}\Lambda^k\gamma_{k,t}g_{\langle\rho_1\rho_2}\cdots g_{\rho_{2k-1}\rho_{2k}}\nabla_{\rho_{2k+1}}\cdots\nabla_{\rho_t}h_{\nu(s-1)\rangle\mu}\ .\end{aligned} \tag{3.31}$$

Here the brackets [ ] denote the largest integer smaller or equal to the argument. The coefficients $\gamma_{k,t}$ are determined by eq. (B.5) in appendix B.

3.3 Bianchi identities

After having solved the torsion constraints, we are now going to discuss some of its consequences, specifically the generalized Bianchi identities. In form language (3.8) reads

$$\begin{aligned}T^{a(s-1),b(t)} \;&=\; \bar{D}\omega^{a(s-1),b(t)} + t(s-t+1)\Lambda\omega^{\langle a(s-1),b(t-1)}\wedge\bar{e}^{b_t\rangle}\\ &\quad+\omega^{a(s-1),b(t)c}\wedge\bar{e}_c \;=\; 0\ ,\\ R^{a(s-1),b(s-1)} \;&=\; \bar{D}\omega^{a(s-1),b(s-1)} + 2(s-1)\Lambda\omega^{\langle a(s-1),b(s-2)}\wedge\bar{e}^{b_{s-1}\rangle}\ .\end{aligned} \tag{3.32}$$

Let us assume that we have solved the first $t-1$ torsion constraints. By acting with $\bar{D}$ on $T^{a(s-1),b(t-1)}$ we can conclude

$$0 \;=\; \bar{D}T^{a(s-1),b(t-1)} \;=\; T^{a(s-1),b(t-1)c} \wedge \bar{e}_c \;, \tag{3.33}$$

where we also used $T^{a(s-1),b(t-2)}=0$. This implies that prior to solving the torsion constraint at level t, the Bianchi identities for the previous ones yield already a nontrivial relation. In components it reads

$$T_{[\mu\nu|\rho(s-1),\sigma(t-1)|\lambda]} \;=\; 0 \;, \tag{3.34}$$

where all indices are curved. Rewriting T in terms of Ω and using the symmetries of the HS connections in the frame indices, one concludes

$$\Omega^{[ab|c(s-1),d(t-1)|e]} \;=\; 0 \;, \tag{3.35}$$

which was required in order to solve the torsion constraint at level t. Finally, applying $\bar{D}$ to the last torsion constraint $t=s-2$, we derive similarly an algebraic Bianchi identity for the Riemann tensor,

$$R_{[\mu\nu|\rho]\sigma(s-2),\lambda(s-1)} \;=\; 0 \;. \tag{3.36}$$

Let us now turn to the differential Bianchi identities of the Riemann tensor. Acting again with $\bar{D}$ and using the AdS relation $\bar{R}^{ab}=-\Lambda\bar{e}^a\wedge\bar{e}^b$ we conclude

$$\bar{D}R^{a(s-1),b(s-1)} \;=\; 0 \;, \tag{3.37}$$

which reads in components with curved incdices

$$\nabla_\mu R_{\nu\rho|\sigma(s-1),\lambda(s-1)} + \nabla_\nu R_{\rho\mu|\sigma(s-1),\lambda(s-1)} + \nabla_\rho R_{\mu\nu|\sigma(s-1),\lambda(s-1)} \;=\; 0 \;. \tag{3.38}$$

Finally this gives rise to a contracted Bianchi identity: Introducing a Ricci tensor

$$(\mathrm{Ric})_{\mu\nu|\rho(s-1),\sigma(s-3)} \;=\; R_{\mu\nu|\rho(s-1),\sigma(s-3)}{}^{\lambda}{}_{\lambda} \;, \tag{3.39}$$

and an analogue of the curvature 'scalar'

$$S_{\mu|\nu(s-2),\rho(s-3)} \;=\; g^{\sigma\lambda}(\mathrm{Ric})_{\sigma\mu|\lambda\nu(s-2),\rho(s-3)} \;, \tag{3.40}$$

one can define the 'Einstein tensor'

$$G_{\nu\rho|\mu\sigma(s-2),\lambda(s-3)} \;=\; (\mathrm{Ric})_{\nu\rho|\mu\sigma(s-2),\lambda(s-3)} - 2g_{\mu[\nu}S_{\rho]|\sigma(s-2),\lambda(s-3)} \;. \tag{3.41}$$

This has been defined such that it satisfied the conservation law

$$\nabla^\mu G_{\nu\rho|\mu\sigma(s-2),\lambda(s-3)} \;=\; 0 \;, \tag{3.42}$$

which follows by contracting (3.38) with $g^{\mu\sigma_1}$ and $g^{\lambda_{s-2}\lambda_{s-1}}$.

3.4 Generalized metricity condition

Let us now relate the HS connections determined above to the de Wit-Freedman connections reviewed in section 2.2. For this we first introduce generalized Christoffel symbols in complete analogy to the spin-2 case (see section 2.1). More specifically, we want to add connection terms to the partial derivative $\partial_\mu e_\nu{}^{a_1\cdots a_{s-1}}$ such that the resulting expression is HS invariant. We first note that for the antisymmetric part in μ,ν we know already the answer, namely the required connections are precisely the HS connections $\omega_\mu{}^{a_1\cdots a_{s-1},b}$ above, as these combine to the HS invariant torsion tensors (3.8). To define the symmetric part as well, we simply introduce as in the spin-2 case a symmetric Christoffel symbol which does the job. Explicitly we define $\widehat{\Gamma}_{\rho(s-1),\mu\nu}$ through the generalized metricity condition

$$\mathcal{D}_\mu h_{\nu\rho(s-1)} \;=\; \partial_\mu h_{\nu\rho(s-1)} + \omega_{\mu|\rho(s-1),\nu} - \widehat{\Gamma}^{(1)}_{\rho(s-1)|\mu\nu} \;=\; 0\;, \tag{3.43}$$

which defines a HS covariant derivative $\mathcal{D}_\mu$. These Christoffel symbols have a different index structure than the de Wit-Freedman connections, and accordingly they have a different transformation behaviour. The latter is determined by (3.43) as follows

$$\delta_\epsilon\widehat{\Gamma}^{(1)}_{\rho(s-1)|\mu\nu}=\frac{2}{s}\partial_\mu\partial_\nu\epsilon_{\rho(s-1)}+\frac{2(s-2)}{s}\partial_{(\mu}\partial_{(\rho_1}\epsilon_{\rho_2\cdots\rho_{s-1})\nu)}-\frac{s-2}{s}\partial_{(\rho_1}\partial_{\rho_2}\epsilon_{\rho_3\cdots\rho_{s-1})\mu\nu}\;. \tag{3.44}$$

This allows to express these connections in terms of the de Wit-Freedman connections as

$$\widehat{\Gamma}^{(1)}_{\rho(s-1)|\mu\nu} \;=\; \Gamma^{(1)}_{(\rho_1,\rho_2\cdots\rho_{s-1})\mu\nu} - \frac{s-2}{2(s-1)}\left(\Gamma^{(1)}_{\mu,\rho(s-1)\nu}+\Gamma^{(1)}_{\nu,\rho(s-1)\mu}\right)\;, \tag{3.45}$$

which can be verified by explicit evaluation. We see that both types of generalized Christoffel connections coincide for spin $s=2$. $\widehat{\Gamma}$ can be written in a closed form as

$$\widehat{\Gamma}_{\rho(s-1)|\mu\nu} \;=\; \partial_\mu h_{\nu\rho(s-1)} + \partial_\nu h_{\rho(s-1)\mu} - \partial_{(\rho_1}h_{\rho_2\cdots\rho_{s-1})\mu\nu}\;, \tag{3.46}$$

which is the obvious generalization of the spin-2 expression (2.12).

The metricity relation can be directly extended to all higher connections and for AdS, by imposing

$$\mathcal{D}_\mu\omega_{\nu|\rho(s-1),\sigma(t)} \;=\; \nabla_\mu\omega_{\nu|\rho(s-1),\sigma(t)} + \omega_{\mu|\rho(s-1),\sigma(t)\nu} - \widehat{\Gamma}^{(t)}_{\rho(s-1),\sigma(t)|\mu\nu} \;=\; 0\;. \tag{3.47}$$

After gauge-fixing the physical HS field $h_{\mu_1\cdots\mu_s}$ to the completely symmetric part, these equations define a hierarchy of generalized Christoffel symbols, which are metric-like t-derivative objects in h. Nevertheless they are different from the de Wit-Freedman connections, though they can be expressed in terms of them. However, since these relations are not relevant for our present analysis, we refrain from computing them here explicitly for general t.

4. Higher-spin dynamics

In this section we will analyze the free HS dynamics in the geometrical frame formalism developed so far. Specifically, we will show that the Einstein-like equations, stating the vanishing of the HS Ricci tensor, are equivalent to the Frønsdal equations in the unconstrained compensator formulation for Minkowski space as well as for AdS. Finally, we discuss the possibility of a geometrical action principle.

4.1 Generalized Einstein equations on Minkowski

We are going to prove that the HS Einstein equation

$$(\mathrm{Ric})_{\mu\nu|\rho(s-1),\sigma(s-3)} \;=\; R_{\mu\nu|\rho(s-1),\sigma(s-3)}{}^{\lambda}{}_{\lambda} = 0\,, \tag{4.1}$$

is equivalent to the Frønsdal formulation. The equation (4.1) appears in the unconstrained Vasiliev equations [36], but more recently also through a Chern-Simons action principle [37]. The former equations are first order differential equations in the so-called unfolded formulation (see [51, 52] and [53]) and read in the linearization [36, 14]

$$\mathcal{F}^{a(s-1),b(t)} \;=\; \delta_{t,s-1}\,\bar{e}_c \wedge \bar{e}_d\, C^{a(s-1)c,b(s-1)d}\,. \tag{4.2}$$

Here $\mathcal{F}$ denotes the (trace-full) HS field strength defined in section 3 and $C^{a(s),b(s)}$ is the so-called Weyl zero-form. The latter is traceless and so (4.2) implies the dynamical equation (4.1) together with the torsion constraints discussed in section 3.2. The same dynamical equation has been derived from the Chern-Simons action principle of [37], upon imposing the torsion constraints. (A more detailed discussion about the possibility of an action principle giving rise to (4.1) will be presented below.)

To explain the approach, we start with the low spin cases $s = 3$ and $s = 4$, improving the discussion of [36, 37]. The explicit form of the spin-3 connections can be read off from (3.26). Inserting into the HS Riemann tensor implies

$$(\mathrm{Ric})_{\mu\nu|\rho\sigma} \;=\; 2\partial_{[\mu}\mathcal{F}_{\nu]\rho\sigma} \;=\; 0\,, \tag{4.3}$$

where the first equation follows by explicit evaluation. This is the so-called Damour-Deser identity, which has first been derived for spin-3 geometry in metric-like formulation in [33].[6] This relation in turn allows to locally integrate by virtue of the Poincaré lemma, resulting in $\mathcal{F}_{\mu\nu\rho} = \partial_\mu\alpha_{\nu\rho}$. Since the Frønsdal operator is completely symmetric, the right-hand side has to be totally symmetric as well. As it still has to satisfy the vanishing curl condition (4.3), this is only possible if $\alpha_{\nu\rho} = \partial_\nu\partial_\rho\alpha$, i.e. if

$$\mathcal{F}_{\mu\nu\rho} \;=\; \partial_\mu\partial_\nu\partial_\rho\alpha\,. \tag{4.4}$$

To conclude, the zero-curl equation (4.3) can be integrated, giving rise to (4.4). These are precisely the compensator equations of Francia and Sagnotti [15, 16], with α being the so-called compensator field. They are invariant under all unconstrained HS transformations by virtue of the compensating transformation

$$\delta_\epsilon\alpha \;=\; \epsilon'\,. \tag{4.5}$$

The appearance of the compensator should not come as a surprise since we started with a HS Ricci tensor being invariant under unconstrained HS transformations. To recover the constrained Frønsdal formulation one simply uses the fact that ϵ' acts as a shift symmetry

[6] For a flat-space generalization to fields in mixed Young tableaux see [18].

on α in order to set the latter to zero. Thus the dynamical content of (4.3), in spite of being a $3^{\rm rd}$ order differential equation, is precisely that of a spin-3 mode on Minkowski space.

The way we presented the derivation of (4.4) was entirely based on the 'classical' Poincaré lemma. There is, however, a more direct way thanks to the extended Poincaré lemma of Dubois-Violette and Henneaux [54, 55] (see also [23, 56]), which applies to tensor fields in more general Young tableaux than the completely antisymmetric ones of differential forms. To be more precise, for a sequence of Young diagrams with projectors $\mathbb{P}_p$ one can define a differential $\tilde{d}$ as follows

$$\tilde{d} \;=\; \mathbb{P}_{p+1} \;\circ\; \partial\,. \tag{4.6}$$

It maps a tensor of degree p to a tensor of degree $p+1$ by first taking the derivative and then symmetrizing according to the Young symmetries encoded by $\mathbb{P}_{p+1}$. If N is the maximal number of columns carried by the Young diagrams, we have $\tilde{d}^{N+1} = 0$. This reduces to the familiar $d^2 = 0$ for differential forms, i.e. for completely antisymmetric tableaux with one column. In the case of a spin-3 field, equation (4.3) can then schematically be written as

$$\tilde{d}\mathcal{F} \;=\; (\mathbb{P}_{\yng(3,1)} \;\circ\; \partial)\mathcal{F} \;=\; 0\,. \tag{4.7}$$

Here we have used the fact that the left-hand side of (4.3) is in $\yng(3,1)$, but interpreted in the antisymmetric basis, in agreement with the Bianchi identity $({\rm Ric})_{[\mu\nu|\rho]\sigma} = 0$. (See the discussion in appendix C of [37].) Since the number of columns is three, we have $\tilde{d}^4 = 0$, and so (4.7) implies by the Poincaré lemma $\mathcal{F} = \tilde{d}^3\alpha$, whose component form coincides with (4.4). In this language, the Frønsdal operator $\mathcal{F}$ is a closed and therefore exact form with respect to a generalized differential.

Let us now turn to the spin-4 case, which shows some new features. The required top spin-4 connection determining the Riemann tensor can be read off from (3.26). Its trace part, which enters the Ricci tensor, can be written as

$$\begin{aligned}\omega_{\mu|\nu(3),\rho}{}^{\sigma}{}_{\sigma} \;&=\partial_\rho\Box h_{\nu(3)\mu} - \partial_{(\nu_1}\Box h_{\nu_2\nu_3)\rho\mu} + 2\partial_{(\nu_1}\partial_{\nu_2}\partial\cdot h_{\nu_3)\rho\mu}\\ &\quad - 2\partial_\rho\partial_{(\nu_1}\partial\cdot h_{\nu_2\nu_3)\mu} + \partial_\rho\partial_{(\nu_1}\partial_{\nu_2}h'_{\nu_3)\mu} - \partial^3_{\nu(3)}h'_{\rho\mu}\,.\end{aligned} \tag{4.8}$$

Upon insertion this yields

$$({\rm Ric})_{\mu\nu|\rho(3),\sigma} \;=\; 4\partial_{\underbracket{\mu}\,}\underbracket{\partial_{[\sigma}\mathcal{F}_{(\rho_1]\rho_2\rho_3)\nu}} \;=\; 0\,, \tag{4.9}$$

where it is understood that the symmetrization has to be performed at the very end. This can again be solved more conveniently by use of the calculus of Dubois-Violette-Henneaux. In this language (4.9) is equivalent to

$$\tilde{d}^2\mathcal{F} \;=\; 0\,, \tag{4.10}$$

where the sequence of Young diagrams consists of $\yng(4,1)$ and $\yng(4,2)$. We have $\tilde{d}^5 = 0$ and so by the Poincaré lemma $\mathcal{F} = \tilde{d}^3\alpha$. This is precisely the compensator equation for spin-4,

$$\mathcal{F}_{\mu\nu\rho\sigma} \;=\; 3\partial_{(\mu}\partial_\nu\partial_\rho\alpha_{\sigma)}\,. \tag{4.11}$$

As before, this equation is invariant under unconstrained HS transformations by virtue of the shift symmetry $\delta_\epsilon \alpha_\mu = \epsilon'_\mu$, which in turn can be used to set the compensator to zero. But, there is actually a novelty as compared to the spin-3 case, since all fields with spin $s \geq 4$ can have a double trace part $h''_{\mu_4 \cdots \mu_s}$. The latter enters as follows: Taking the divergence of the Frønsdal operator and its trace, say in the spin-4 case, results in the Bianchi identity [16]

$$\partial \cdot \mathcal{F}_{\mu\nu\rho} - \tfrac{3}{2}\partial_{(\mu}\mathcal{F}'_{\nu\rho)} \;=\; -\tfrac{3}{2}\partial_\mu \partial_\nu \partial_\rho h'' \,. \tag{4.12}$$

This in turn implies that (4.11) yields as a consistency condition the additional equation

$$h'' \;=\; \partial \cdot \alpha \,. \tag{4.13}$$

Therefore, due to the Bianchi identity (4.12), gauge-fixing α_μ to zero, sets at the same time the double-trace part of h to zero. Thus one recovers indeed the Frønsdal equations together with the constraints (2.16), which was required to consistently describe the spin-4 field.

In the remainder of this section we will turn to the general spin-s case. The Damour-Deser identity, which was crucial for the above derivation, generalizes as follows: The frame-like spin-s Ricci tensor is given by

$$\begin{aligned}(\mathrm{Ric})_{\mu\nu|\rho(s-1),\sigma(s-3)} \;&=\; \frac{2s}{3}\mathbb{P}_{(s-1,s-3)}\partial_{\underbrace{\mu \partial_{\sigma(s-3)}\mathcal{F}_{\rho(s-1)}\nu}} \\ &=\; 2^{s-2}\mathbb{P}_{\rho(s-1)}\mathbb{P}_{\sigma(s-3)}\partial_{[\mu}\partial_{[\sigma_1}\cdots \partial_{[\sigma_{s-3}}\mathcal{F}_{\nu]\rho_1]\cdots \rho_{s-3}]\rho_{s-2}\rho_{s-1}}\,,\end{aligned} \tag{4.14}$$

where in the second line we have written out the projector explicitly in order to make the curl-like structure manifest. It is understood that we perform the antisymmetrizations over (μ,ν), (σ_1,ρ_1), etc. This identity can be proven as follows: We first notice that it has the $(s-1,s-3)$ Young symmetry required by the left-hand side. Second, it is an HS invariant s-th derivative object in the physical spin-s field, since $\delta_\epsilon \mathcal{F}_{\mu_1\cdots\mu_s} \sim \partial_{(\mu_1}\partial_{\mu_2}\partial_{\mu_3}\epsilon'_{\mu_4\cdots\mu_s)}$. So up to a factor, there is nothing else one can write down. Finally the factor originates from the $s-2$ antisymmetrizations. Using this identity, the HS Einstein equation can be integrated as above to the Frønsdal equations in the compensator formulation, which upon gauge-fixing sets the double-trace part to zero. We finally note that the presented derivation of the Frønsdal equations coincides with the one given in [17], despite the fact that we were starting in a frame-like formalism.

4.2 Generalized Einstein equations on AdS

Let us now turn to the HS dynamics on an AdS background. The unconstrained *frame-like* formulation has so far been discussed only in case of a spin-3 field in [37]. In harmony with the above derivation in flat space, the spin-3 case can be analyzed as follows. Inserting (3.30) into the Ricci tensor and making repeated use of the defining AdS relation for covariant derivatives,

$$[\nabla_\mu, \nabla_\nu]V_\rho \;=\; \Lambda\left(g_{\nu\rho}V_\mu - g_{\mu\rho}V_\nu\right)\,, \tag{4.15}$$

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one finds that the Einstein equations can be written as

$$(\mathrm{Ric})_{\mu\nu|\rho\sigma} \;=\; R_{\mu\nu|\rho\sigma,}{}^{\lambda}{}_{\lambda} \;=\; 2\nabla_{[\mu}\mathcal{F}^{\mathrm{AdS}}_{\nu]\rho\sigma} \;=\; 0\,. \tag{4.16}$$

Thus we see that the Damour-Deser identity directly generalizes to a spin-3 field on AdS. In principle, to systematically solve this equation would require a refinement of the cohomological analysis of Dubois-Violette and Henneaux to AdS. Here we will rather follow a more pragmatic route by simply showing that it can be locally solved in the required way. Indeed, one finds that

$$\mathcal{F}^{\mathrm{AdS}}_{\mu\nu\rho} \;=\; \nabla_{(\mu}\nabla_{\nu}\nabla_{\rho)}\alpha - 4\Lambda g_{(\mu\nu}\nabla_{\rho)}\alpha \tag{4.17}$$

solves (4.16). Note that this is a solution only due to the term proportional to the cosmological constant. The appearance of this additional term comes as no surprise, since the right-hand side has to take exactly the compensator form [16], which in turn is fixed by gauge invariance: In fact, as above we have to set $\delta_\epsilon\alpha = \epsilon'$ in order for (4.17) to be invariant under the unconstrained HS symmetry.

Next we turn to the spin-4 case, which is slightly more involved due to the following reason. In section 3.2 we have seen that the HS connections on AdS depend on the cosmological constant, whose powers increase with the spin, see eq. (3.31). For a spin-4 field this means that the top connection depends linearly on Λ, which upon insertion gives rise to a Ricci tensor with a quadratic dependence. This in turn implies that the naive AdS covariantization of the Damour-Deser identity cannot be correct, since it depends, through the mass-like term, only linearly on Λ. Therefore, the Damour-Deser identity receives a correction, which is proportional to Λ. One may compute this by explicit evaluation, but this becomes rather tedious. Fortunately, it is possible to fix this form by general reasoning as follows: Since the leading term proportional to $\nabla^2\mathcal{F}$ is separately invariant under constrained HS transformations with $\epsilon' = 0$, it follows that the correction term has to be invariant as well. Thus, since it is 2^{nd} order in derivatives (as one can see from (3.31)), it has to be the Frønsdal operator. Finally, it remains to determine the relative coefficient. This can be done by requiring gauge invariance or by using the fact that the left-hand side is known to satisfy the Bianchi identity (3.38). One finds

$$(\mathrm{Ric})_{\mu\nu|\rho(3),\sigma} = \frac{8}{3}\mathbb{P}_{(3,1)}\left(\nabla_{\underline{\mu}}\underline{\nabla_\sigma\mathcal{F}_{\rho(3)\nu}} - \Lambda g_{\underline{\mu\sigma}}\underline{\mathcal{F}_{\rho(3)\nu}}\right)\,. \tag{4.18}$$

In case of general spin we find the following expression

$$\begin{aligned}(\mathrm{Ric})_{\mu\nu|\rho(s-1),\sigma(s-3)} =&\frac{2s}{3}\mathbb{P}_{(s-1,s-3)}\Big(\nabla_{\underline{\mu}}\underline{\nabla_{\sigma_1}\cdots\nabla_{\sigma_{s-3}}\mathcal{F}_{\rho(s-1)\nu}}\\ &+\sum_{k=1}^{[\frac{s-2}{2}]}\vartheta^s_k\Lambda^k g_{\sigma_1\sigma_2}\cdots g_{\sigma_{2k-3}\sigma_{2k-2}}g_{\underline{\mu}\sigma_{2k-1}}\underline{\nabla_{\sigma_{2k}}\cdots\nabla_{\sigma_{s-3}}\mathcal{F}_{\rho(s-1)\nu}}\Big),\end{aligned} \tag{4.19}$$

where ϑ^s_k are coefficients determined in appendix C to be

$$\vartheta^s_k \;=\; (-1)^k\tau_{2k-1}\frac{(s-3)!}{(s-2k-2)!}\,, \tag{4.20}$$

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with τ_{2k-1} the Taylor coefficients in $\tan x = \sum_{k=1}^{\infty} \tau_{2k-1} x^{2k-1}$.

After we have shown that the Ricci tensor satisfies a Damour-Deser identity for general spin also on AdS, we are now able to recover the Frønsdal formulation. Since the resulting expression for the Ricci tensor is gauge invariant, we can immediately conclude – even without inspecting the precise coefficients — that the compensator ansatz

$$\mathcal{F}_{\mu_1\cdots\mu_s} \;=\; \tfrac{1}{2}(s-1)(s-2)\big(\nabla_{(\mu_1}\nabla_{\mu_2}\nabla_{\mu_3}\alpha_{\mu_4\cdots\mu_s)} - 4\Lambda g_{(\mu_1\mu_2}\nabla_{\mu_3}\alpha_{\mu_4\cdots\mu_s)}\big) \qquad (4.21)$$

solves the HS Einstein equation (4.1), as the right-hand side takes precisely the form of the gauge variation of $\mathcal{F}$ under ϵ', see (2.19).

4.3 Towards an action principle?

In this section we will briefly discuss the possibility of a geometrical action principle which gives rise to the Einstein equations (4.1) in the frame formalism.[7] More precisely, we are asking if there is an action with the following properties:

(i) It should be written entirely in terms of differential forms, in analogy to the vielbein formulation of general relativity.

(ii) It ought to be manifestly invariant under all (unconstrained) HS transformations.

(iii) It should imply the torsion constraints together with the Einstein equations in a 1[st] order formalism.

(iv) Finally it should be local and not contain inverse powers of differential operators.[8]

Ideally one would like to satisfy requirement (*iii*) in a way that allows a 1.5 order formalism. This would mean that the torsion constraint determining, say, $\omega^{a(s-1),b(t)}$ should result from varying precisely this field, while the Einstein-like equation should be obtained by varying with respect to the physical HS field $e^{a(s-1)}$. In turn this would allow to consistently eliminate the HS connections at the level of the action, without altering the HS invariance or the Einstein equations, simply due to the fact that any additional variation of the HS connections resulting from this elimination would be proportional to the vanishing torsion. Unfortunately, such a formulation cannot exist, since the Einstein equations for spin s have $2s-2$ free indices in contrast to the s indices of the physical field to be varied (which coincide only for $s=2$). So the best one can hope for is a formulation which yields the required equations by varying with respect to some other fields.

In order to proceed we first note that condition (*i*) in combination with (*ii*) implies that the action has to be written entirely in terms of the Lorentz covariant HS field strengths defined in section 3. To be specific, let us discuss the case of a spin-3 field on a $D=4$ Minkowski space, which we expect to exhibit generic features. We have to construct a 4-form out of the HS curvatures, starting with T^{ab}. A term like $\bar{e}^a\wedge\bar{e}^b\wedge T_{ab}$ vanishes identically due to the symmetry of T^{ab}. So the only possible terms seem to be those that contract HS

[7] We are grateful to Misha Vasiliev for discussions on this point.

[8] For non-local actions in the context of unconstrained frame fields see [18].

curvatures among themselves. Defining the Ricci 2-form $R_{ab} = \frac{1}{2}(\mathrm{Ric})_{\mu\nu|ab}dx^\mu \wedge dx^\nu$ one can write

$$S_{\rm HS} \;=\; \int T^{ab} \wedge T_{ab} + T^{ab,c} \wedge T_{ab,c} + T^{ab} \wedge R_{ab} \;, \tag{4.22}$$

which is manifestly HS invariant. Up to the relative coefficients, this is the unique expression, which is quadratic in spin-3 curvatures and carries trace parts only on the Riemann tensor. It turns out that the dynamical content of (4.22) is to a large extent independent on the precise value of these coefficients. The equations of motion obtained from (4.22) by varying with respect to e^{ab}, $\omega^{ab,c}$ and $\omega^{ab,cd}$, respectively, read

$$dT^{ab} \;=\; 0 \;, \tag{4.23}$$

$$2T^{\langle ab} \wedge \bar{e}^{c\rangle} - 2dT^{ab,c} + R^{\langle ab} \wedge \bar{e}^{c\rangle} \;=\; 0 \;, \tag{4.24}$$

$$2T^{\langle ab,c} \wedge \bar{e}^{d\rangle} - dT^{\langle ab}\eta^{cd\rangle} \;=\; 0 \;. \tag{4.25}$$

Inserting (4.23) into (4.25) implies

$$T^{\langle ab,c} \wedge \bar{e}^{d\rangle} \;=\; 0 \qquad \Longleftrightarrow \qquad T^{ab,c} \;=\; 0 \;, \tag{4.26}$$

i.e. we correctly recover one of the torsion constraints. Unfortunately, for the other torsion constraint the equations of motion (4.23) imply only the weaker condition $T^{ab} = d\xi^{ab}$, i.e. the torsion tensor is exact, but not necessarily zero. However, focusing on the particular solution where it is zero yields with (4.24) the Einstein equations (4.1). We conclude that, while it seems to be impossible along these lines to construct a manifestly HS invariant action which satisfies the above requirements (*i*)–(*iv*), it is nevertheless possible to define an action which at least contains a consistently propagating spin-3 mode, potentially with dynamical torsion.

5. Conclusions and discussion

In this paper we presented a systematic analysis of the frame or vielbein formulation of unconstrained HS fields. We determined the torsion and Riemann curvature tensor for arbitrary spin-s fields. Imposing vanishing torsion allowed us to express the HS connections in terms of the physical field, whose solutions were determined explicitly in flat space and AdS. The corresponding HS Christoffel symbols related via a generalized metricity condition have been discussed. We would like to stress that these are not identical to the de Wit-Freedman connections, though they are of course related, as we have seen in section 3.4. (So, referring to the generalized spin-connections like in [36, 37] as de Wit-Freedman connections is slightly misleading.) However, the de Wit-Freedman connections are, in contrast, not known on AdS in a closed form and have been determined perturbatively in the inverse AdS length only recently [46]. Finally, we analyzed the HS dynamics in this unconstrained formulation. As previously seen in the metric-like formulation in flat space [17, 36], we found that the higher-derivative Einstein equations satisfy the Damour-Deser identity and can therefore be locally integrated to the Frønsdal equations in the compensator formulation. Moreover, we derived a generalization of the Damour-Deser identity for AdS, which in turn showed that also the AdS-Frønsdal equations can be encoded in the s-derivative Einstein equations. This confirms in particular that the appearance of 'extra fields' at the free-field level of the Chern-Simons action in [37] is not in conflict with the requirement of standard physical field equations of 2^{nd} derivative order. In other words, it verifies that the higher derivatives, which naturally appear in HS theories, are gauge artefacts.

We close with a few general comments on the advantages of this unconstrained formalism, but also of its disadvantages, as compared to the original one of [19]. First of all, it is attractive since it parallels very closely the spin-2 case of gravity in vielbein form, as there is a clear distinction between torsion and curvature tensors, while the latter gives rise to manifestly HS invariant Einstein equations. Apart from that, since we are ultimately interested in the coupling of HS fields to gravity, the trace constraints that are inevitable in the conventional frame formulation seem to be unnatural. Finally, since the higher derivatives do appear anyway at the interacting level of the HS theories of [34, 35], it seems to be legitimate to admit them also at the free-field level, especially since they allow to nicely recover the Frønsdal formulation.

However, there are also disadvantages if one wants to encode the dynamics in an action principle instead of equations of motion. Specifically, as we discussed in section 4.3, due to the mismatch of free indices it is impossible to construct an action which yields the Einstein equations by varying with respect to the physical HS field. Instead, these field equations are obtained by varying with respect to one of the HS connections. Moreover, it seems to be difficult, if not impossible, to obtain all torsion constraints from an action. These are definitely shortcomings, since if one believes that only the HS field $h_{\mu_1\cdots\mu_s}$ is of physical significance, there should be an action principle formulated in terms of this field only. This is in contrast to the constrained formulation of [19]. For this, a 1^{st} order action can be given that contains only a single connection, which in turn can be eliminated by its equations of motion, giving rise to the standard 2^{nd} order Frønsdal action. However, its HS invariance is not manifest. In particular, without the presence of the corresponding gauge field, the single connection is subject to one of the Stückelberg shift symmetries discussed in section 3, which is possible only due to the tracelessness conditions. Altogether this can be interpreted in the sense that, if one insists on a manifestly HS invariant, unconstrained action principle, the hierarchy of HS connections are not mere auxiliary fields, but instead have to carry their own dynamics. In other words, this indicates a propagating torsion as in the Chern-Simons theories of [37]. While from this point of view there are certain unconventional features of an unconstrained formalism, we think that it nevertheless possesses a number of attractive properties which deserve further investigations.

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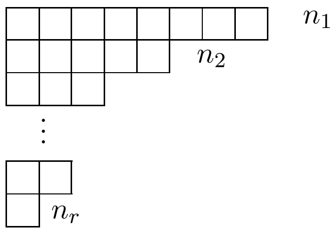
A. Notation and conventions

Throughout the article, we work in D dimensions so that the space-time indices run between $\mu, \nu, \ldots = 0, 1, \ldots, D-1$. Sometimes we utilize form language and use for a p-form $F_p = \frac{1}{p!} F_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$. $A = 0, 1, \ldots, D-1, 0'$ denote $\mathrm{SO}(D-1,2)$ vector indices, out of which the first D indices $a = 0, 1, \ldots, D-1$ are Lorentz indices.

We use in the main text the language of Young tableaux, for which our conventions are as follows: In case of AdS or Lorentz tensors they encode the irreducible representations of $GL(D+1)$ or $GL(D)$, respectively, since we are working with trace-full tensors. We employ the symmetric basis with the convention that to impose the Young-tableau symmetry of a diagram like



we first antisymmetrize over the columns and then symmetrize the row indices. A Young diagram with n_k boxes in row k is denoted by $(n_1, n_2, \ldots, n_r)$. Specifically, in order to impose these symmetries on a tensor $X_{a^1_1 \cdots a^1_{n_1} | a^2_1 \cdots a^2_{n_2} | \cdots | a^r_1 \cdots a^r_{n_r}}$, we first antisymmetrize in $a^1_k a^2_k \cdots a^r_k$ for $k = 1, \ldots, n_r$, and then symmetrize in $a^k_1 \cdots a^k_{n_k}$ for $k = 1, \ldots, r$.[9] In particular, the action of a 2-row (m,n) Young projector $\mathbb{P}_{(m,n)}$ on a tensor $X_{a_1 \cdots a_m | b_1 \cdots b_n}$ with no a priori symmetries in the two sets of indices reads $(n \leq m)$

$$\begin{aligned} \mathbb{P}_{(m,n)} X_{a_1 \cdots a_m | b_1 \cdots b_n} &= X_{\langle a_1 \cdots a_m | b_1 \cdots b_n \rangle} \\ &= \frac{m-n+1}{m+1} \mathbb{P}_{(m)} \mathbb{P}_{(n)} \Big(X_{a_1 \cdots a_m | b_1 \cdots b_n} - X_{b_1 a_2 \cdots a_m | a_1 b_2 \cdots b_n} \\ &\qquad + \cdots + (-1)^n X_{b_1 \cdots b_n a_1 \cdots a_m | a_1 \cdots a_n} \Big) , \end{aligned} \tag{A.1}$$

where $\mathbb{P}_{(m)}$ and $\mathbb{P}_{(n)}$ imposes total (unit-weight) symmetrization in the two sets of indices. Here the overall normalization has been chosen such that the projector satisfies $\mathbb{P}^2 = \mathbb{P}$.

B. Recursion relations for AdS connections

In this appendix we derive the expression (3.31) for the HS connections in terms of the physical fields in AdS. We are working in the gauge-fixed formalism described in section 3.2.

We start by recalling that the solution of $\omega_{\mu | \nu(s-1), \rho(t)}$ in terms of $\Omega_{\rho_t \mu | \nu(s-1), \rho(t-1)}$ in (3.23) is valid also in AdS since the identities (3.35) still hold after gauge fixing, as proven

[9] We use a different convention to impose Young-tableau symmetry than in [37].

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in section 3.3. For $t = 1$ the latter may be expressed in terms of the physical field $h_{\mu(s)}$ and we immediately deduce the expression for the $t = 1$ connection:

$$\omega_{\mu|\nu(s-1),\rho} = -\frac{s}{s-1}\Omega_{\mu\langle\rho|\nu(s-1)\rangle} = \frac{s}{s-1}\nabla_{\langle\rho}h_{\nu(s-1)\rangle\mu}\,, \tag{B.1}$$

whose explicit form is given in (3.29). To determine the $t = 2$ connection we first have to express $\Omega_{\rho_2\mu|\nu(s-1),\rho_1}$ in terms of the physical field by using (3.13) and the expression for the $t = 1$ connection just found, and then to insert this into (3.23) with $t = 2$. The result for Ω is

$$\Omega_{\rho_2\mu|\nu(s-1),\rho_1} = \mathbb{P}_{(s-1,1)}\Big(\frac{s}{s-1}\nabla_{\rho_2}\nabla_{\rho_1}h_{\mu\nu(s-1)} + \Lambda s\; h_{\nu(s-1)\rho_2}g_{\mu\rho_1}\Big) - (\rho_2 \leftrightarrow \mu)\,. \tag{B.2}$$

By noting that the terms in (B.2) involving $h_{\nu(s-1)\rho_2}$ vanish identically under the $\mathbb{P}_{(s-1,2)}$ projector, we recover the expression in (3.30).

By using the formulas (3.13) and (3.23) one can easily show that in general (with $1 \le t \le s-1$) we have that

$$\begin{aligned}\omega_{\mu|\nu(s-1),\rho(t)} = \frac{s-t+1}{s-t}\Big(&\nabla_{\langle\rho_t}\omega_{\mu|\nu(s-1),\rho(t-1)\rangle} \\ &-\Lambda(t-1)(s-t+2)\omega_{\mu|\langle\nu(s-1),\rho(t-2)}g_{\rho_{t-1}\rho_t\rangle}\Big) - (\rho_t \leftrightarrow \mu)\,.\end{aligned} \tag{B.3}$$

Here we have used that the $(s-1, t-1)$ projector can be suppressed under the $(s-1, t)$ projector. To solve for $\omega_{\mu|\nu(s-1),\rho(t)}$ in terms of the physical field h, we solve this equation iteratively by inserting the expressions for $\omega_{\mu|\nu(s-1),\rho(t-1)}(h)$ and $\omega_{\mu|\nu(s-1),\rho(t-2)}(h)$ which are assumed to have been solved for in previous iteration steps. (We remark that the $(\rho_t \leftrightarrow \mu)$ terms in (B.3) vanish under the projector $\mathbb{P}_{(s-1,t)}$ after gauge-fixing h to the completely symmetric part, since then the ω are in irreducible (s, t) tableaux.) For instance, the $t = 3$ connection can be solved for by inserting the expressions for the $t = 1$ and $t = 2$ connections which are displayed in (3.29) and (3.30), respectively. Hence, this defines a recursive problem which is solvable by induction.

The general solution is found to be given by

$$\omega_{\mu|\nu(s-1),\rho(t)} = \sum_{k=0}^{[t/2]}\Lambda^k\gamma_{k,t}g_{\langle\rho_1\rho_2}\cdots g_{\rho_{2k-1}\rho_{2k}}\nabla_{\rho_{2k+1}}\cdots\nabla_{\rho_t}h_{\nu(s-1)\rangle\mu}\,, \tag{B.4}$$

where we have included the leading term in (3.31) as the $k = 0$ contribution. The coefficients $\gamma_{k,t}$ with $0 \le k \le [t/2]$ are determined by the recursive relation

$$\gamma_{k,t} = \frac{s-t+1}{s-t}\Big(\gamma_{k,t-1} - (t-1)(s-t+2)\gamma_{k-1,t-2}\Big)\,, \tag{B.5}$$

with the 'initial conditions'

$$\gamma_{0,0} = 1\,, \qquad \gamma_{1,2} = -\frac{s(s-1)}{s-2}\,. \tag{B.6}$$

Here it is understood that $\gamma_{k,t} = 0$ for $t < 0$. Eq. (B.5) can be derived by inserting the expression (B.4) into both sides of (B.3) and comparing the corresponding powers of Λ.

The explicit form of the first coefficients is

$$\gamma_{0,t} = \frac{s}{s-t}\,, \tag{B.7}$$

$$\gamma_{1,t} = \frac{st(t-1)\big(2t-1-3s\big)}{6(s-t)}\,, \tag{B.8}$$

$$\gamma_{2,t} = \frac{st(t-1)(t-2)(t-3)\big(45s^2-60(t-1)s+4t(5t-12)+7\big)}{360(s-t)}\,. \tag{B.9}$$

For even $t \geq 4$ we find an expression for the $k = t/2$ coefficient

$$\gamma_{t/2,t} = (-1)^{t/2}\frac{(t-1)(t-3)(s-t-1)(s-t+3)(s-t+4)}{s-t}\,. \tag{B.10}$$

C. Damour-Deser identity in AdS

In this appendix we will derive the Damour-Deser identity (4.19) in AdS. The result follows from two requirements: (i) that the relation involves an expansion in powers of Λ over terms which have a linear dependence on the Frønsdal operator (see the discussion above (4.18)); and (ii) that the Bianchi identity

$$\nabla_{[\lambda}(\mathrm{Ric})_{\mu\nu]|\rho(s-1),\sigma(s-3)} \;=\; 0 \tag{C.1}$$

following from (3.38) holds. Note that (i) implies that the identity reduces to the flat-space relation (4.14) for $\Lambda \to 0$.

The requirement (i) above implies that the expansion is given by a sum over terms of the form $(\Lambda g)^m \nabla^n \mathcal{F}$, where g is the metric. These terms have to be compatible with dimensional analysis and Young-tableau symmetry. Thus, the most general ansatz which is compatible with the requirements (i) and (ii) above takes the form

$$\begin{aligned}(\mathrm{Ric})_{\mu\nu|\rho(s-1),\sigma(s-3)} =& \frac{2s}{3}\mathbb{P}_{(s-1,s-3)}\Big(\nabla_{\underline{\mu}}\nabla_{\sigma_1}\cdots\nabla_{\sigma_{s-3}}\mathcal{F}_{\rho(s-1)\underline{\nu}} \\ &+ \sum_{k=1}^{[\frac{s-2}{2}]}\vartheta^s_k\Lambda^k g_{\sigma_1\sigma_2}\cdots g_{\sigma_{2k-3}\sigma_{2k-2}}g_{\underline{\mu}\sigma_{2k-1}}\nabla_{\sigma_{2k}}\cdots\nabla_{\sigma_{s-3}}\mathcal{F}_{\rho(s-1)\underline{\nu}}\Big).\end{aligned} \tag{C.2}$$

The challenge is to determine the coefficients ϑ^s_k. Note that by this ansatz the algebraic Bianchi identity (3.36) is identically satisfied and does not lead to any further constraint.

Let us illustrate the method by fixing the first two coefficients ϑ^s_1 and ϑ^s_2. To simplify the presentation we use the notation $\nabla_{\mu_1\mu_2\cdots\mu_n} \equiv \nabla_{\mu_1}\nabla_{\mu_2}\cdots\nabla_{\mu_n}$, with the indices arranged in this particular order. We focus on terms proportional to Λ in (C.1), which are given by

$$\begin{aligned}\mathbb{P}_{(s-1,s-3)}\Big(&[\nabla_\lambda,\nabla_\mu]\nabla_{\sigma_1\cdots\sigma_{s-3}}\mathcal{F}_{\rho(s-1)\nu} + [\nabla_\mu,\nabla_\nu]\nabla_{\sigma_1\cdots\sigma_{s-3}}\mathcal{F}_{\rho(s-1)\lambda} \\ &+ [\nabla_\nu,\nabla_\lambda]\nabla_{\sigma_1\cdots\sigma_{s-3}}\mathcal{F}_{\rho(s-1)\mu} + 3!\vartheta^s_1\Lambda g_{\sigma_{s-3}[\mu}\nabla_{\lambda|\sigma(s-4)|}\mathcal{F}_{\nu]\rho(s-1)}\Big)\,.\end{aligned} \tag{C.3}$$

In order to determine ϑ^s_1 it is sufficient to inspect terms of a specific structure, say, $g_{\mu\sigma_{s-3}}\nabla_{\sigma(s-4)}$. Using (4.15), these can be written as

$$\Lambda\mathbb{P}_{\rho(s-1)}\mathbb{P}_{\sigma(s-3)}g_{\mu\sigma_{s-3}}\Big(\big(\nabla_{\lambda\sigma(s-4)}+\cdots+\nabla_{\sigma(s-4)\lambda}\big)\mathcal{F}_{\rho(s-1)\nu}+\vartheta^s_1\nabla_{\lambda\sigma(s-4)}\mathcal{F}_{\rho(s-1)\nu}\Big). \tag{C.4}$$

When performing the $(s-1, s-3)$ projection, we were allowed to ignore the antisymmetrization contained in it, since they would give rise to different index structures. Next we have to commute the covariant derivatives such that they all appear in the form $\nabla_{\lambda\sigma(s-4)}$. One finds

$$\begin{aligned}\Lambda\mathbb{P}_{\rho(s-1)}\mathbb{P}_{\sigma(s-3)}g_{\mu\sigma_{s-3}}\Big((\vartheta_1^s+(s-3))\nabla_{\lambda\sigma(s-4)}\mathcal{F}_{\rho(s-1)\nu}+(s-4)[\nabla_{\sigma_{s-4}},\nabla_\lambda]\nabla_{\sigma(s-5)}\\ +(s-5)\nabla_{\sigma_{s-4}}[\nabla_{\sigma_{s-5}},\nabla_\lambda]\nabla_{\sigma(s-6)}+\dots+\nabla_{\sigma(s-5)}[\nabla_{\sigma_{s-4}},\nabla_\lambda]\Big)\mathcal{F}_{\rho(s-1)\nu}\ .\end{aligned} \tag{C.5}$$

Therefore we conclude

$$\vartheta_1^s \;=\; -(s-3)\ , \tag{C.6}$$

while the commutators give rise to terms that are quadratic in Λ and therefore have to be cancelled by higher-order contributions. As a consistency check we note that the terms proportional to $g_{\mu\rho_{s-1}}$, $g_{\mu\nu}$ and $g_{\mu\lambda}$ vanish identically, which follows from the total symmetry of the Frønsdal operator.

Let us now turn to the next, i.e., quadratic order. For this we evaluate the commutators in (C.5) and include the $k=2$ term in (C.2), to arrive at an analogous expression to (C.4):

$$\begin{aligned}-\Lambda^2\mathbb{P}_{\rho(s-1)}\mathbb{P}_{\sigma(s-3)}g_{\mu\sigma_{s-3}}g_{\sigma_{s-5}\sigma_{s-4}}\Big(\big(\chi_{s-4}\nabla_{\lambda\sigma(s-6)}+\dots+\chi_2\nabla_{\sigma(s-6)\lambda}\big)\mathcal{F}_{\rho(s-1)\nu}\\ +\vartheta_2^s\nabla_{\lambda\sigma(s-6)}\mathcal{F}_{\rho(s-1)\nu}\Big)\ ,\end{aligned} \tag{C.7}$$

where we have defined

$$\chi_m \;:=\; \sum_{n=m}^{s-4} n \;=\; \tfrac{1}{2}(s-3-m)(s-4+m)\ . \tag{C.8}$$

Then focusing on terms with the index structure $g_{\mu\sigma}g_{\sigma\sigma}$ determines the next coefficient to be

$$\vartheta_2^s \;=\; \sum_{m=2}^{s-4}\chi_m \;=\; \tfrac{1}{3}(s-3)(s-4)(s-5)\ . \tag{C.9}$$

The general case proceeds in exact analogy. By repeating the steps just described and focusing on a specific index structure, we ultimately obtain the simple form displayed in (4.20). As a consistency check, we verified up to cubic order in Λ that all other terms cancel.

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